SOF

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1 Introduction

This document contains some mathematical derivations for the Separated Oscillatory Field (SOF) model.

2 Derivation of the SOF model

2.1 Ramsey

This part is based on the paper by Ramsey, 1950, Phys Rev 78, 695.
Eq (13) is

$$2b = g\mu_0 H_1/\hbar$$

b is the transition dipole moment and is real.
Based on definitions in Eq (5)

$$\cos(\theta) = (\omega_0 - \omega)/a, \sin(\theta) = 2b/a$$

$$a = [(\omega_0 - \omega)^2 + (2b)^2]^{\frac{1}{2}}, \omega_0 = (W_q - W_p)/\hbar$$

$\cos(\theta), \sin(\theta)$, and a are real, given that ω and ω_0 are real physical quantities.
Eq (10) is

$$C_q(2\tau + T) = \boxed{2i\sin(\theta)} \cdot \backslash$$

$$[\cos(\theta)\sin^2(\frac{1}{2}a\tau)\sin(\frac{1}{2}\lambda T) - \frac{1}{2}\sin(a\tau)\cos(\frac{1}{2}\lambda T)] \cdot \backslash$$

$$\boxed{\exp(-i...)}$$

where

$$\lambda = (\overline{W_q} - \overline{W_p})/\hbar - \omega$$

To find probability,

$$P_{p,q} = |C_q|^2 = C_q^* \cdot C_q$$

For generic complex number z, $\overline{z \cdot z} = \overline{z} \cdot \overline{z}$.
Therefore,

$$\boxed{|2i\sin(\theta)|^2 = 4\sin^2(\theta)}$$

$$\boxed{|\exp(-i...)|^2} = 1$$

For the following term,

$$\cos(\theta)\sin^2(\frac{1}{2}a\tau)\sin(\frac{1}{2}\lambda T) - \frac{1}{2}\sin(a\tau)\cos(\frac{1}{2}\lambda T)$$

By using double angle formula $\sin(2x) = \frac{1}{2}\sin(x)\cos(x)$,

$$= \cos(\theta)\sin^2(\frac{1}{2}a\tau)\sin(\frac{1}{2}\lambda T) - \sin(\frac{1}{2}a\tau)\cos(\frac{1}{2}a\tau)\cos(\frac{1}{2}\lambda T)$$

$$= \sin(\frac{1}{2}a\tau)[\cos(\theta)\sin(\frac{1}{2}a\tau)\sin(\frac{1}{2}\lambda T) - \cos(\frac{1}{2}a\tau)\cos(\frac{1}{2}\lambda T)]$$

Thus,

$$|\cos(\theta)\sin^2(\frac{1}{2}a\tau)\sin(\frac{1}{2}\lambda T) - \frac{1}{2}\sin(a\tau)\cos(\frac{1}{2}\lambda T)|^2$$

$$= \sin^2(\frac{1}{2}a\tau)[\cos(\theta)\sin(\frac{1}{2}a\tau)\sin(\frac{1}{2}\lambda T) - \cos(\frac{1}{2}a\tau)\cos(\frac{1}{2}\lambda T)]^2$$

$$= \sin^2(\frac{1}{2}a\tau)[\cos(\frac{1}{2}\lambda T)\cos(\frac{1}{2}a\tau) - \cos(\theta)\sin(\frac{1}{2}\lambda T)\sin(\frac{1}{2}a\tau)]^2$$

Finally,

$$P_{p,q} = |C_q|^2$$

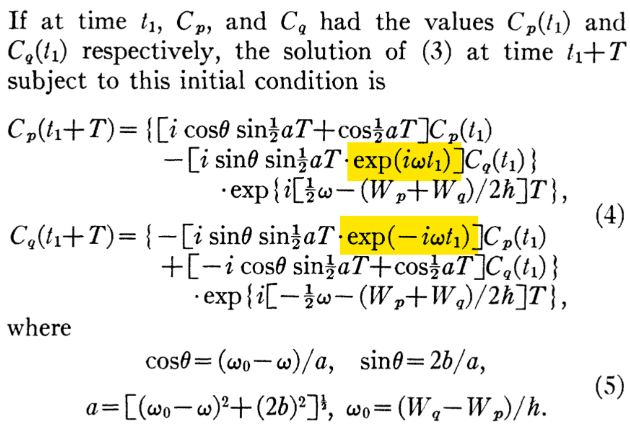
$$= 4\sin^2(\theta) \cdot \sin^2(\frac{1}{2}a\tau)[\cos(\frac{1}{2}\lambda T)\cos(\frac{1}{2}a\tau) - \cos(\theta)\sin(\frac{1}{2}\lambda T)\sin(\frac{1}{2}a\tau)]^2$$

2.2 Lamb shift

This part is based on Resonance-Narrowed-Lamb-Shift Measurement in Hydrogen, n=3.

2.2.1 Theory of line shape

Eq 9a and 9b are not enough to derive Eq 19a and 19b. From Ramsey's paper, A Molecular Beam Resonance Method with Separated Oscillating Fields, we should expect an exponential decay term in the solution.



If at time t_1, C_p, and C_q had the values $C_p(t_1)$ and $C_q(t_1)$ respectively, the solution of (3) at time t_1+T subject to this initial condition is

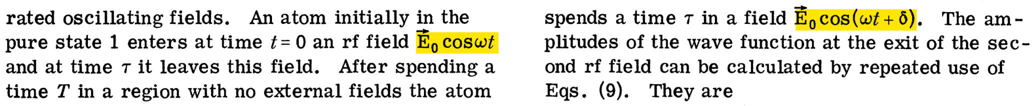
$$\begin{aligned} C_p(t_1+T) = \{&[i\cos\theta\sin\tfrac{1}{2}aT + \cos\tfrac{1}{2}aT]C_p(t_1) \\ &- [i\sin\theta\sin\tfrac{1}{2}aT\cdot\exp(i\omega t_1)]C_q(t_1)\} \\ &\cdot\exp\{i[\tfrac{1}{2}\omega - (W_p+W_q)/2\hbar]T\}, \\ C_q(t_1+T) = \{&-[i\sin\theta\sin\tfrac{1}{2}aT\cdot\exp(-i\omega t_1)]C_p(t_1) \\ &+ [-i\cos\theta\sin\tfrac{1}{2}aT + \cos\tfrac{1}{2}aT]C_q(t_1)\} \\ &\cdot\exp\{i[-\tfrac{1}{2}\omega - (W_p+W_q)/2\hbar]T\}, \end{aligned} \quad (4)$$

where

$$\cos\theta = (\omega_0-\omega)/a, \quad \sin\theta = 2b/a,$$
$$a = [(\omega_0-\omega)^2 + (2b)^2]^{\frac{1}{2}}, \quad \omega_0 = (W_q - W_p)/\hbar. \quad (5)$$

Figure 1: Eq 4 from Ramsey's paper, with exponential decay term highlighted.

Also, since the positroniums are passing through two waveguides with a phase delay δ. In the equation given in Eq 16a and 16b, we need to set $\delta = 0$ to derive Eq 19a and 19b.



rated oscillating fields. An atom initially in the pure state 1 enters at time $t=0$ an rf field $\vec{E}_0\cos\omega t$ and at time τ it leaves this field. After spending a time T in a region with no external fields the atom spends a time τ in a field $\vec{E}_0\cos(\omega t+\delta)$. The amplitudes of the wave function at the exit of the second rf field can be calculated by repeated use of Eqs. (9). They are

Figure 2: Initial condition from Resonance-Narrowed-Lamb-Shift

2.2.2 Final probability for state 1

Analytical derivation
Some handy math identities:

$$\overline{zz} = \overline{z} \cdot \overline{z} \tag{2.2.1}$$

$$(z)^2 + (\overline{z})^2 = (x+iy)^2 + (x-iy)^2 = 2x^2 - 2y^2 = 2(\Re(z)^2 - \Im(z)^2) \tag{2.2.2}$$

$$(z)^2 - (\overline{z})^2 = (x+iy)^2 - (x-iy)^2 = 4ixy = 4i\Re(z)\Im(z) \tag{2.2.3}$$

$$e^{ix} = \cos(x) + i\sin(x) \tag{2.2.4}$$

Assuming c_1 and c_2 are correct.
From equations 10-15,

$$\Omega = \omega - \omega_0, \omega_0 = \omega_1 - \omega_2$$

$$Q = \frac{1}{2}(\gamma_1 - \gamma_2)$$

$$a = \left[4V^2 + (\Omega + iQ)^2\right]^{\frac{1}{2}}$$

$$\sin\theta = \frac{2V}{a}$$

$$\cos\theta = \frac{(\Omega + iQ)}{a}$$

Given that Ω, Q, and V are real, a, $\sin\theta$, and $\cos\theta$ are complex.
Staring from Eq 19a.

$$\begin{aligned} c_1(\tau + T + \tau) = {} & \exp(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau - \tfrac{1}{2}\gamma_1 T - i(\omega+\omega_1+\omega_2)\tau - i\omega_1 T) \\ & \times\left\{\left[\cos(\tfrac{1}{2}a\tau) + i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\right]^2 - \exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T - i(\delta+\Omega T)\right]\sin^2(\theta)\sin^2(\tfrac{1}{2}a\tau)\right\} \end{aligned} \tag{2.2.5}$$

Take $\left[\cos(\frac{1}{2}a\tau) + i\cos(\theta)\sin(\frac{1}{2}a\tau)\right]^2$ out from bracket.

$$\begin{aligned} c_1(\tau + T + \tau) = {} & \exp(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau - \tfrac{1}{2}\gamma_1 T - i(\omega+\omega_1+\omega_2)\tau - i\omega_1 T) \\ & \times\left[\cos(\tfrac{1}{2}a\tau) + i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\right]^2 \\ & \times\left\{1 - \exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T - i(\delta+\Omega T)\right]\frac{\sin^2(\theta)\sin^2(\frac{1}{2}a\tau)}{\left[\cos(\frac{1}{2}a\tau)+i\cos(\theta)\sin(\frac{1}{2}a\tau)\right]^2}\right\} \end{aligned} \tag{2.2.6}$$

Let

$$z = \frac{\sin(\theta)\sin(\frac{1}{2}a\tau)}{\cos(\frac{1}{2}a\tau) + i\cos(\theta)\sin(\frac{1}{2}a\tau)} \tag{2.2.7}$$

Then

$$\begin{aligned} c_1(\tau + T + \tau) = {} & \exp(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau - \tfrac{1}{2}\gamma_1 T - i(\omega+\omega_1+\omega_2)\tau - i\omega_1 T) \\ & \times\left[\cos(\tfrac{1}{2}a\tau) + i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\right]^2 \\ & \times\left\{1 - \exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T - i(\delta+\Omega T)\right]z^2\right\} \end{aligned} \tag{2.2.8}$$

Take modulus squared by using Eq 2.2.1.

$$\begin{aligned} |c_1(\tau + T + \tau)|^2 = {} & \left|\exp(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau - \tfrac{1}{2}\gamma_1 T - i(\omega+\omega_1+\omega_2)\tau - i\omega_1 T)\right|^2 \\ & \times\left|\left[\cos(\tfrac{1}{2}a\tau) + i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\right]^2\right|^2 \\ & \times\left|1 - \exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T - i(\delta+\Omega T)\right]z^2\right|^2 \end{aligned} \tag{2.2.9}$$

Calculate each term.

$$
\begin{aligned}
&\left|\exp(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau-\tfrac{1}{2}\gamma_1 T-i(\omega+\omega_1+\omega_2)\tau-i\omega_1 T)\right|^2\\
&\quad=\exp(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau-\tfrac{1}{2}\gamma_1 T+i(\omega+\omega_1+\omega_2)\tau+i\omega_1 T)\\
&\qquad\times\exp(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau-\tfrac{1}{2}\gamma_1 T-i(\omega+\omega_1+\omega_2)\tau-i\omega_1 T)\\
&\quad=\exp(2\times(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau-\tfrac{1}{2}\gamma_1 T))
\end{aligned}
\tag{2.2.10}
$$

$$
\begin{aligned}
&\left|\left[\cos(\tfrac{1}{2}a\tau)+i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\right]^2\right|^2\\
&\quad=\left[\left|\cos(\tfrac{1}{2}a\tau)+i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\right|^2\right]^2
\end{aligned}
\tag{2.2.11}
$$

$$
\begin{aligned}
&\left|1-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T-i(\delta+\Omega T)\right]z^2\right|^2\\
&\quad=\left[1-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T+i(\delta+\Omega T)\right]\overline{z}^2\right]\\
&\qquad\times\left[1-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T-i(\delta+\Omega T)\right]z^2\right]\\
&\quad=1-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T+i(\delta+\Omega T)\right]\overline{z}^2-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T-i(\delta+\Omega T)\right]z^2\\
&\qquad+\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T+i(\delta+\Omega T)\right]\overline{z}^2\cdot\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T-i(\delta+\Omega T)\right]z^2\\
&\quad=1-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T+i(\delta+\Omega T)\right]\overline{z}^2-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T-i(\delta+\Omega T)\right]z^2\\
&\qquad+\exp\left[(\gamma_1-\gamma_2)T)\right]\overline{z}^2\left[|z|^2\right]^2\\
&\quad=1+\exp\left[(\gamma_1-\gamma_2)T)\right]\left[|z|^2\right]^2\\
&\qquad-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T+i(\delta+\Omega T)\right]\overline{z}^2-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T-i(\delta+\Omega T)\right]z^2
\end{aligned}
\tag{2.2.12}
$$

Note that $1+\exp\left[(\gamma_1-\gamma_2)T)\right]\left[|z|^2\right]^2$ has no dependence on δ.

$$
\begin{aligned}
&-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T+i(\delta+\Omega T)\right]\overline{z}^2-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T-i(\delta+\Omega T)\right]z^2\\
&\quad=-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T\right]\left[\exp\left[i(\delta+\Omega T)\right]\overline{z}^2+\exp\left[-i(\delta+\Omega T)\right]z^2\right]\\
&\quad=-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T\right]\\
&\qquad\left[\left[\cos(\delta+\Omega T)+i\sin(\delta+\Omega T)\right]\overline{z}^2+\left[\cos(-(\delta+\Omega T))+i\sin(-(\delta+\Omega T))\right]z^2\right]\\
&\quad=-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T\right]\\
&\qquad\left[\left[\cos(\delta+\Omega T)+i\sin(\delta+\Omega T)\right]\overline{z}^2+\left[\cos(\delta+\Omega T)-i\sin(\delta+\Omega T)\right]z^2\right]\\
&\quad=-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T\right]\left[\cos(\delta+\Omega T)\left[\overline{z}^2+z^2\right]+i\sin(\delta+\Omega T)\left[\overline{z}^2-z^2\right]\right]
\end{aligned}
\tag{2.2.13}
$$

By using Eq 2.2.2 and Eq 2.2.3, we have

$$
\begin{aligned}
&-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T\right]\left[\cos(\delta+\Omega T)\left[\overline{z}^2+z^2\right]+i\sin(\delta+\Omega T)\left[\overline{z}^2-z^2\right]\right]\\
&=-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T\right]\left[\cos(\delta+\Omega T)2\left[\Re(z)^2-\Im(z)^2\right]+i\sin(\delta+\Omega T)\left[4i\Re(z)\Im(z)\right]\right]\\
&=-2\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T\right]\left[\cos(\delta+\Omega T)\left[\Re(z)^2-\Im(z)^2\right]-\sin(\delta+\Omega T)\left[2\Re(z)\Im(z)\right]\right]
\end{aligned}
\tag{2.2.14}
$$

Collecting Eq 2.2.14, Eq 2.2.13 and Eq 2.2.12, we have

$$
\begin{aligned}
&\left|1-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T-i(\delta+\Omega T)\right]z^2\right|^2\\
&\quad=1+\exp\left[(\gamma_1-\gamma_2)T)\right]\left[|z|^2\right]^2\\
&\qquad-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T+i(\delta+\Omega T)\right]\overline{z}^2-\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T-i(\delta+\Omega T)\right]z^2\\
&\quad=1+\exp\left[(\gamma_1-\gamma_2)T)\right]\left[|z|^2\right]^2\\
&\qquad-2\exp\left[\tfrac{1}{2}(\gamma_1-\gamma_2)T\right]\left[\cos(\delta+\Omega T)\left[\Re(z)^2-\Im(z)^2\right]-\sin(\delta+\Omega T)\left[2\Re(z)\Im(z)\right]\right]
\end{aligned}
\tag{2.2.15}
$$

Collecting Eq 2.2.10, Eq 2.2.11 and Eq 2.2.15, we have

$$
\begin{aligned}
&|c_1(\tau+T+\tau)|^2 = \\
&\quad \exp(2\times(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau-\tfrac{1}{2}\gamma_1 T)) \\
&\quad \times\Big[\big|\cos(\tfrac{1}{2}a\tau)+i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\big|^2\Big]^2 \\
&\quad \times\Bigg\{1+\exp\big[(\gamma_1-\gamma_2)T)\big]\big[|z|^2\big]^2 \\
&\quad -2\exp\big[\tfrac{1}{2}(\gamma_1-\gamma_2)T\big]\Big[\cos(\delta+\Omega T)\big[\Re(z)^2-\Im(z)^2\big]-\sin(\delta+\Omega T)\big[2\Re(z)\Im(z)\big]\Big]\Bigg\}
\end{aligned}
\tag{2.2.16}
$$

Now, we will calculate signal $S=|c_1|^2_{\delta=\pi}-|c_1|^2_{\delta=0}$

$$
\begin{aligned}
&|c_1|^2_{\delta=\pi}-|c_1|^2_{\delta=0} \\
&\quad = \exp(2\times(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau-\tfrac{1}{2}\gamma_1 T)) \\
&\qquad \times\Big[\big|\cos(\tfrac{1}{2}a\tau)+i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\big|^2\Big]^2 \\
&\qquad \times\Bigg\{-2\exp\big[\tfrac{1}{2}(\gamma_1-\gamma_2)T\big]\Big[\cos(\pi+\Omega T)\big[\Re(z)^2-\Im(z)^2\big]-\sin(\pi+\Omega T)\big[2\Re(z)\Im(z)\big]\Big] \\
&\qquad +2\exp\big[\tfrac{1}{2}(\gamma_1-\gamma_2)T\big]\Big[\cos(\Omega T)\big[\Re(z)^2-\Im(z)^2\big]-\sin(\Omega T)\big[2\Re(z)\Im(z)\big]\Big]\Bigg\}
\end{aligned}
\tag{2.2.17}
$$

As $\cos(x+\pi)=-\cos(x)$ and $\sin(x+\pi)=-\sin(x)$

$$
\begin{aligned}
&|c_1|^2_{\delta=\pi}-|c_1|^2_{\delta=0} \\
&\quad = \exp(2\times(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau-\tfrac{1}{2}\gamma_1 T)) \\
&\qquad \times\Big[\big|\cos(\tfrac{1}{2}a\tau)+i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\big|^2\Big]^2 \\
&\qquad \times\Bigg\{4\exp\big[\tfrac{1}{2}(\gamma_1-\gamma_2)T\big]\Big[\cos(\Omega T)\big[\Re(z)^2-\Im(z)^2\big]-\sin(\Omega T)\big[2\Re(z)\Im(z)\big]\Big]\Bigg\} \\
&\quad = 4\exp\big[2\times(-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau-\tfrac{1}{2}\gamma_1 T)+\tfrac{1}{2}(\gamma_1-\gamma_2)T\big] \\
&\qquad \times\Big[\big|\cos(\tfrac{1}{2}a\tau)+i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\big|^2\Big]^2 \\
&\qquad \times\Big[\cos(\Omega T)\big[\Re(z)^2-\Im(z)^2\big]-\sin(\Omega T)\big[2\Re(z)\Im(z)\big]\Big] \\
&\quad = 4\exp\big[-\tfrac{1}{2}(\gamma_1+\gamma_2)2\tau-\gamma_1 T+\tfrac{1}{2}(\gamma_1-\gamma_2)T\big] \\
&\qquad \times\Big[\big|\cos(\tfrac{1}{2}a\tau)+i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\big|^2\Big]^2\Big[\cos(\Omega T)\big[\Re(z)^2-\Im(z)^2\big]-\sin(\Omega T)\big[2\Re(z)\Im(z)\big]\Big] \\
&\quad = 4\exp\big[-\tfrac{1}{2}(\gamma_1+\gamma_2)2\tau-\tfrac{1}{2}(\gamma_1+\gamma_2)T\big] \\
&\qquad \times\Big[\big|\cos(\tfrac{1}{2}a\tau)+i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\big|^2\Big]^2\Big[\cos(\Omega T)\big[\Re(z)^2-\Im(z)^2\big]-\sin(\Omega T)\big[2\Re(z)\Im(z)\big]\Big] \\
&\quad = 4\exp\big[-\tfrac{1}{2}(\gamma_1+\gamma_2)(2\tau+T)\big] \\
&\qquad \times\Big[\big|\cos(\tfrac{1}{2}a\tau)+i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\big|^2\Big]^2\Big[\cos(\Omega T)\big[\Re(z)^2-\Im(z)^2\big]-\sin(\Omega T)\big[2\Re(z)\Im(z)\big]\Big]
\end{aligned}
\tag{2.2.18}
$$

As z is defined in 2.2.7

$$
|z|^2=\left|\frac{\sin(\theta)\sin(\frac{1}{2}a\tau)}{\cos(\frac{1}{2}a\tau)+i\cos(\theta)\sin(\frac{1}{2}a\tau)}\right|^2 \tag{2.2.19}
$$

$$
\begin{aligned}
&|c_1|^2_{\delta=\pi}-|c_1|^2_{\delta=0} \\
&\quad = 4\exp\big[-\tfrac{1}{2}(\gamma_1+\gamma_2)(2\tau+T)\big] \\
&\qquad \times\big|\cos(\tfrac{1}{2}a\tau)+i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\big|^2\big|\sin(\theta)\sin(\tfrac{1}{2}a\tau)\big|^2 \\
&\qquad \times\tfrac{1}{|z|^2}\Big[\cos(\Omega T)\big[\Re(z)^2-\Im(z)^2\big]-\sin(\Omega T)\big[2\Re(z)\Im(z)\big]\Big] \\
&\quad = 4\exp\big[-\tfrac{1}{2}(\gamma_1+\gamma_2)(2\tau+T)\big] \\
&\qquad \times\big|\cos(\tfrac{1}{2}a\tau)+i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\big|^2\big|\sin(\theta)\sin(\tfrac{1}{2}a\tau)\big|^2 \\
&\qquad \times\Big[\cos(\Omega T)\big[\tfrac{\Re(z)^2-\Im(z)^2}{|z|^2}\big]-\sin(\Omega T)\big[\tfrac{2\Re(z)\Im(z)}{|z|^2}\big]\Big]
\end{aligned}
\tag{2.2.20}
$$

For a complex number z

$$\sin(\arg(z)) = \frac{\Im(z)}{|z|} \tag{2.2.21}$$

$$\cos(\arg(z)) = \frac{\Re(z)}{|z|} \tag{2.2.22}$$

$$\begin{aligned}
&|c_1|^2_{\delta=\pi} - |c_1|^2_{\delta=0} \\
&\quad = 4\exp\left[-\tfrac{1}{2}(\gamma_1+\gamma_2)(2\tau+T)\right] \\
&\qquad \times \left|\cos(\tfrac{1}{2}a\tau) + i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\right|^2 \left|\sin(\theta)\sin(\tfrac{1}{2}a\tau)\right|^2 \\
&\qquad \times \left[\cos(\Omega T)\left[\cos^2(\arg(z)) - \sin^2(\arg(z))\right] - \sin(\Omega T)2\left[\cos(\arg(z))\sin(\arg(z))\right]\right]
\end{aligned} \tag{2.2.23}$$

Using double angle formulae

$$\cos(2\arg(z)) = \cos^2(\arg(z)) - \sin^2(\arg(z)) \tag{2.2.24}$$

$$\sin(2\arg(z)) = 2\cos(\arg(z))\sin(\arg(z)) \tag{2.2.25}$$

Therefore,

$$\begin{aligned}
&|c_1|^2_{\delta=\pi} - |c_1|^2_{\delta=0} \\
&\quad = 4\exp\left[-\tfrac{1}{2}(\gamma_1+\gamma_2)(2\tau+T)\right] \\
&\qquad \times \left|\cos(\tfrac{1}{2}a\tau) + i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\right|^2 \left|\sin(\theta)\sin(\tfrac{1}{2}a\tau)\right|^2 \\
&\qquad \times \left[\cos(\Omega T)\cos(2\arg(z)) - \sin(\Omega T)\sin(2\arg(z))\right]
\end{aligned} \tag{2.2.26}$$

Using cosine subtraction formula.

$$\cos(a-b) = \cos(a)\cos(b) - \sin(a)\sin(b) \tag{2.2.27}$$

$$\begin{aligned}
&|c_1|^2_{\delta=\pi} - |c_1|^2_{\delta=0} \\
&\quad = 4\exp\left[-\tfrac{1}{2}(\gamma_1+\gamma_2)(2\tau+T)\right] \\
&\qquad \times \left|\cos(\tfrac{1}{2}a\tau) + i\cos(\theta)\sin(\tfrac{1}{2}a\tau)\right|^2 \left|\sin(\theta)\sin(\tfrac{1}{2}a\tau)\right|^2 \cos(\Omega T - 2\arg(z))
\end{aligned} \tag{2.2.28}$$

It is highly like that there is a missing factor of 2 in the paper.

Numerical comparison

fig 3_missing_factor_2

July 4, 2023

[ ]:
```python
import numpy as np
import matplotlib.pyplot as plt
```

[ ]:
```python
gamma_1 = 0.063e8 # Hz
gamma_2 = 1.86e8 # Hz

V=1.8e8/2   # Hz
Q = 1/2*(gamma_1-gamma_2)
```

[ ]:
```python
def c_1(Omega, delta, tau, T, V):
    """
    Omega: angular frequency in G rad/s
    delta: phase in rad
    tau: time in ns
    T: time in ns

    gamma_1, gamma_2 in GHz
    """
    a = np.sqrt(4*V**2+(Omega+1j*Q)**2)

    sin_theta = 2*V/a
    cos_theta = (Omega + 1j*Q)/a

    exp_part = np.exp(-1/2*(gamma_1 + gamma_2)*tau-1/2*gamma_1*T)

    x= 1/2*a*tau
    cos_cos_sin_2_part= (np.cos(x)+1j*cos_theta*np.sin(x))**2

    exp_sin_sin_part = np.exp(1/2*(gamma_1-gamma_2)*T - 1j*(delta + Omega*T)) *␣
 ↪\
        sin_theta**2*np.sin(x)**2

    return exp_part*(cos_cos_sin_2_part-exp_sin_sin_part)
```

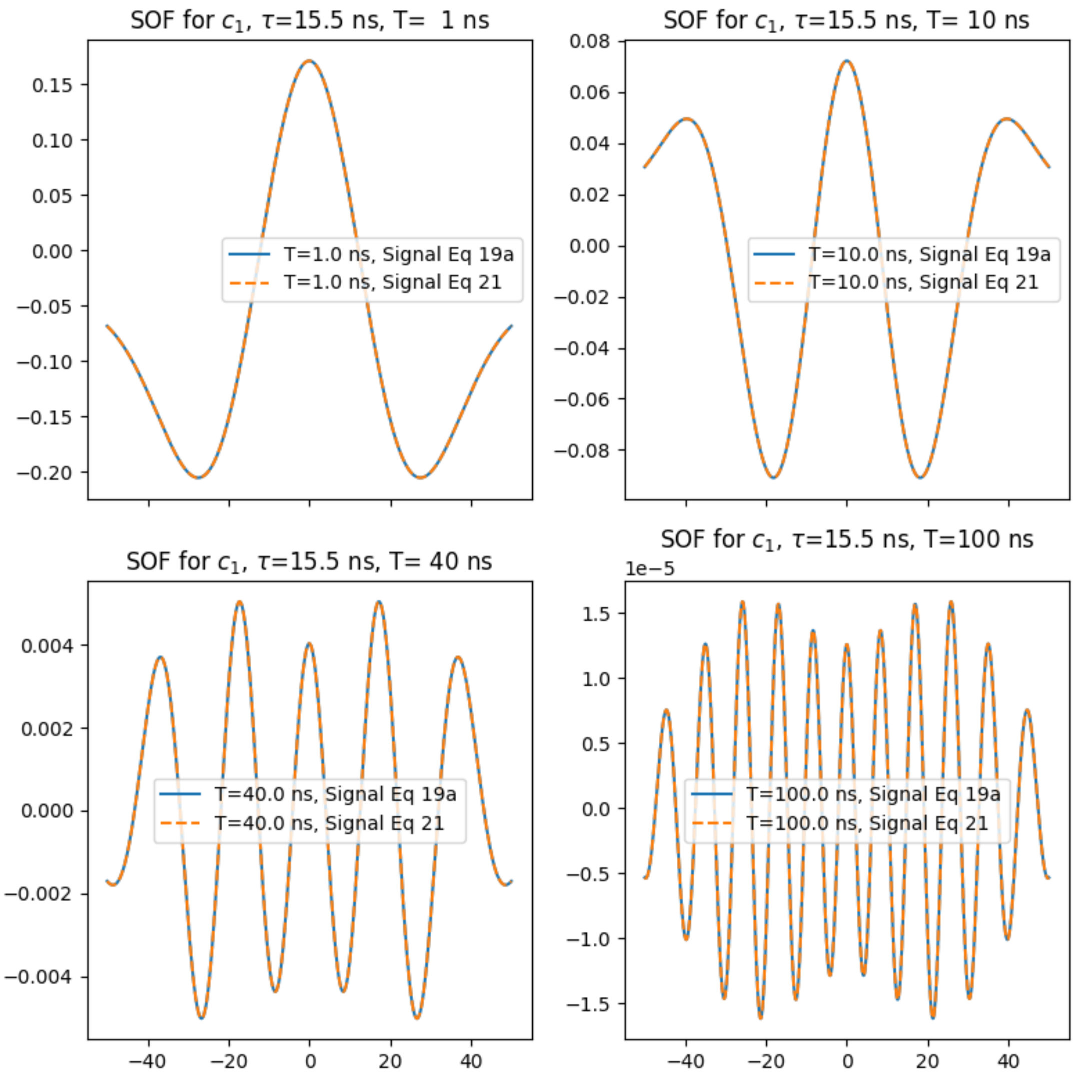
[ ]:
```python
def c_1_modulus_square(Omega, delta, tau, T, V):
    return np.abs(c_1(Omega, delta, tau, T, V))**2
```

```
def signal(Omega, tau, T, V):
    return c_1_modulus_square(Omega, np.pi, tau, T, V) -␣
 ↪c_1_modulus_square(Omega, 0, tau, T, V)
```

```
[ ]: def signal_21(Omega, tau, T, V):
    """
    Omega: angular frequency in G rad/s
    delta: phase in rad
    tau: time in ns
    T: time in ns

    gamma_1, gamma_2 in GHz
    """
    exp_part = 4 * np.exp(-(1/2)*(gamma_1 + gamma_2)*(2*tau+T))
    a = np.sqrt(4*V**2+(Omega+1j*Q)**2)
    x = (1/2)*a * tau

    cos_theta = (Omega + 1j*Q)/a
    sin_theta = 2*V/a

    first_modulus_squared = np.abs(np.cos(x) + 1j*cos_theta*np.sin(x))**2
    second_modulus_squared = np.abs(sin_theta*np.sin(x))**2

    Gamma = 2 * np.angle(sin_theta*np.sin(x) /
                         (np.cos(x)+1j*cos_theta*np.sin(x)))

    return exp_part*first_modulus_squared*second_modulus_squared*np.cos(Omega*T␣
 ↪- Gamma)
```

```
[ ]: tau = 15.5e-9  # s

frequency_range = 50  # MHz
frequncies = np.linspace(-frequency_range, frequency_range, 1000)  # MHz

_T_values = [1e-9, 10e-9, 40e-9, 100e-9]

num_rows = len(_T_values) // 2 + len(_T_values) % 2
num_cols = 2

fig, axs = plt.subplots(num_rows, num_cols, figsize=(8, 8), sharex=True)

for i, _T in enumerate(_T_values):
    row = i // num_cols
    col = i % num_cols
    ax = axs[row, col]
    ax.plot(frequncies, signal(2 * np.pi * frequncies * 1e6, tau, _T, V),␣
 ↪label=f"T={_T * 1e9} ns, Signal Eq 19a")
```

```python
    ax.plot(frequncies, signal_21(2 * np.pi * frequncies * 1e6, tau, _T, V), "--", label=f"T={_T * 1e9} ns, Signal Eq 21")
    ax.legend()
    ax.set_title(f"SOF for $c_1$, $\\tau$={tau * 1e9:3g} ns, T={_T * 1e9:3g} ns")

# Remove empty subplots if there are any
if len(_T_values) < num_rows * num_cols:
    for i in range(len(_T_values), num_rows * num_cols):
        fig.delaxes(axs.flatten()[i])

plt.tight_layout()
plt.show()
```



2.2.3 Final probability for state 2

We can apply the same procedure to the final state 2.

$$\begin{aligned} &|c_2|^2_{\delta=\pi} - |c_2|^2_{\delta=0} \\ &\quad = \exp\left[-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau - \tfrac{1}{2}\gamma_2 T + i(\omega-\omega_1-\omega_2)\tau - i\omega_2 T\right] \\ &\quad\quad \times(-i\sin\theta\sin\tfrac{1}{2}a\tau) \\ &\quad\quad \times\left\{(\cos\tfrac{1}{2}a\tau - i\cos\theta\sin\tfrac{1}{2}a\tau) + \exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T + i(\delta+\Omega T)\right](\cos\tfrac{1}{2}a\tau + i\cos\theta\sin\tfrac{1}{2}a\tau)\right\} \end{aligned} \tag{2.2.29}$$

Take $(\cos\frac{1}{2}a\tau - i\cos\theta\sin\frac{1}{2}a\tau)$ out of the bracket.

$$\begin{aligned} &c_2(\tau+T+\tau) = \\ &\quad \exp\left[-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau - \tfrac{1}{2}\gamma_2 T + i(\omega-\omega_1-\omega_2)\tau - i\omega_2 T\right] \\ &\quad \times(-i\sin\theta\sin\tfrac{1}{2}a\tau)\times(\cos\tfrac{1}{2}a\tau - i\cos\theta\sin\tfrac{1}{2}a\tau) \\ &\quad \times\left\{1+\exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T + i(\delta+\Omega T)\right]\tfrac{\cos\frac{1}{2}a\tau + i\cos\theta\sin\frac{1}{2}a\tau}{\cos\frac{1}{2}a\tau - i\cos\theta\sin\frac{1}{2}a\tau}\right\} \end{aligned} \tag{2.2.30}$$

Let

$$z = \frac{\cos\frac{1}{2}a\tau + i\cos\theta\sin\frac{1}{2}a\tau}{\cos\frac{1}{2}a\tau - i\cos\theta\sin\frac{1}{2}a\tau} \tag{2.2.31}$$

Then

$$\begin{aligned} &c_2(\tau+T+\tau) = \\ &\quad \exp\left[-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau - \tfrac{1}{2}\gamma_2 T + i(\omega-\omega_1-\omega_2)\tau - i\omega_2 T\right] \\ &\quad \times(-i\sin\theta\sin\tfrac{1}{2}a\tau)\times(\cos\tfrac{1}{2}a\tau - i\cos\theta\sin\tfrac{1}{2}a\tau) \\ &\quad \times\left\{1+\exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T + i(\delta+\Omega T)\right]z\right\} \end{aligned} \tag{2.2.32}$$

Take modulus squared

$$\begin{aligned} &|c_2(\tau+T+\tau)|^2 = \\ &\quad |\exp\left[-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau - \tfrac{1}{2}\gamma_2 T + i(\omega-\omega_1-\omega_2)\tau - i\omega_2 T\right]|^2 \\ &\quad \times|\sin\theta\sin\tfrac{1}{2}a\tau|^2 \times |\cos\tfrac{1}{2}a\tau - i\cos\theta\sin\tfrac{1}{2}a\tau|^2 \\ &\quad \times|1+\exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T + i(\delta+\Omega T)\right]z|^2 \end{aligned} \tag{2.2.33}$$

Calculate each term separately.

$$\begin{aligned} &|\exp\left[-\tfrac{1}{2}(\gamma_1+\gamma_2)\tau - \tfrac{1}{2}\gamma_2 T + i(\omega-\omega_1-\omega_2)\tau - i\omega_1 T\right]|^2 \\ &\quad = \exp\left[-(\gamma_1+\gamma_2)\tau - \gamma_2 T\right] \end{aligned} \tag{2.2.34}$$

$$\begin{aligned} &|1+\exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T + i(\delta+\Omega T)\right]z|^2 \\ &\quad = \left[1+\exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T - i(\delta+\Omega T)\right]\overline{z}\right]\left[1+\exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T + i(\delta+\Omega T)\right]z\right] \\ &\quad = 1+\exp\left[(\gamma_2-\gamma_1)T\right]|z|^2 + \exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T - i(\delta+\Omega T)\right]\overline{z} + \exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T + i(\delta+\Omega T)\right]z \end{aligned} \tag{2.2.35}$$

Only $\exp\left[\frac{1}{2}(\gamma_2-\gamma_1)T - i(\delta+\Omega T)\right]\overline{z} + \exp\left[\frac{1}{2}(\gamma_2-\gamma_1)T + i(\delta+\Omega T)\right]z$ depends on δ.

$$\begin{aligned} &\exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T - i(\delta+\Omega T)\right]\overline{z} + \exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T + i(\delta+\Omega T)\right]z \\ &\quad = \exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T\right]\left[\cos(\delta+\Omega T) - i\sin(\delta+\Omega T)\right]\overline{z} + \exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T\right]\left[\cos(\delta+\Omega T) + i\sin(\delta+\Omega T)\right]z \\ &\quad = \exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T\right]\left[\cos(\delta+\Omega T)(\overline{z}+z) - i\sin(\delta+\Omega T)(\overline{z}-z)\right] \end{aligned} \tag{2.2.36}$$

As $z+\overline{z} = 2\Re z$ and $z-\overline{z} = 2i\Im z$, we have

$$\begin{aligned} &\exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T\right]\left[\cos(\delta+\Omega T)(\overline{z}+z) - i\sin(\delta+\Omega T)(\overline{z}-z)\right] \\ &\quad = \exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T\right]\left[\cos(\delta+\Omega T)2\Re z + i\sin(\delta+\Omega T)2i\Im z\right] \\ &\quad = 2\exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T\right]\left[\cos(\delta+\Omega T)\Re z - \sin(\delta+\Omega T)\Im z\right] \\ &\quad = 2\exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T\right]\left[\cos(\delta+\Omega T)\cos(\arg(z))|z| - \sin(\delta+\Omega T)\sin(\arg(z))|z|\right] \\ &\quad = 2\exp\left[\tfrac{1}{2}(\gamma_2-\gamma_1)T\right]|z|\cos\bigl(\arg(z) + (\delta+\Omega T)\bigr) \end{aligned} \tag{2.2.37}$$

Now, we will calculate term $S = |c_2|^2_{\delta=\pi} - |c_2|^2_{\delta=0}$.

$$
\begin{aligned}
S &= |c_2|^2_{\delta=\pi} - |c_2|^2_{\delta=0} \\
&= \exp\big[-(\gamma_1+\gamma_2)\tau - \gamma_2 T\big] \times |\sin\theta \sin \tfrac{1}{2}a\tau|^2 \times |\cos\tfrac{1}{2}a\tau - i\cos\theta\sin\tfrac{1}{2}a\tau|^2 \\
&\quad \times 2\exp\big[\tfrac{1}{2}(\gamma_2-\gamma_1)T\big]|z|\big[\cos\big(\arg(z)+\pi+\Omega T\big) - \cos\big(\arg(z)+\Omega T\big)\big] \\
&= \exp\big[-(\gamma_1+\gamma_2)\tau - \gamma_2 T\big] \times |\sin\theta \sin \tfrac{1}{2}a\tau|^2 \times |\cos\tfrac{1}{2}a\tau - i\cos\theta\sin\tfrac{1}{2}a\tau|^2 \\
&\quad \times 2\exp\big[\tfrac{1}{2}(\gamma_2-\gamma_1)T\big]|z|\big[-\cos\big(\arg(z)+\Omega T\big) - \cos\big(\arg(z)+\Omega T\big)\big] \\
&= -4\exp\big[-(\gamma_1+\gamma_2)\tau - \gamma_2 T\big] \times |\sin\theta \sin \tfrac{1}{2}a\tau|^2 \times |\cos\tfrac{1}{2}a\tau - i\cos\theta\sin\tfrac{1}{2}a\tau|^2 \\
&\quad \times \exp\big[\tfrac{1}{2}(\gamma_2-\gamma_1)T\big]|z|\cos\big(\arg(z)+\Omega T\big) \\
&= -4\exp\big[-\tfrac{1}{2}(\gamma_1+\gamma_2)(2\tau+T)\big] \times |\sin\theta \sin \tfrac{1}{2}a\tau|^2 \times |\cos\tfrac{1}{2}a\tau - i\cos\theta\sin\tfrac{1}{2}a\tau|^2 \\
&\quad \times |z|\cos\big(\arg(z)+\Omega T\big)
\end{aligned}
\tag{2.2.38}
$$

Numerical comparison

c_2_verification

July 4, 2023

[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
```

[ ]:
```
gamma_1 = 0.063e8   # Hz
gamma_2 = 1.86e8   # Hz

V = 1.8e8/2    # Hz
Q = 1/2*(gamma_1-gamma_2)
```

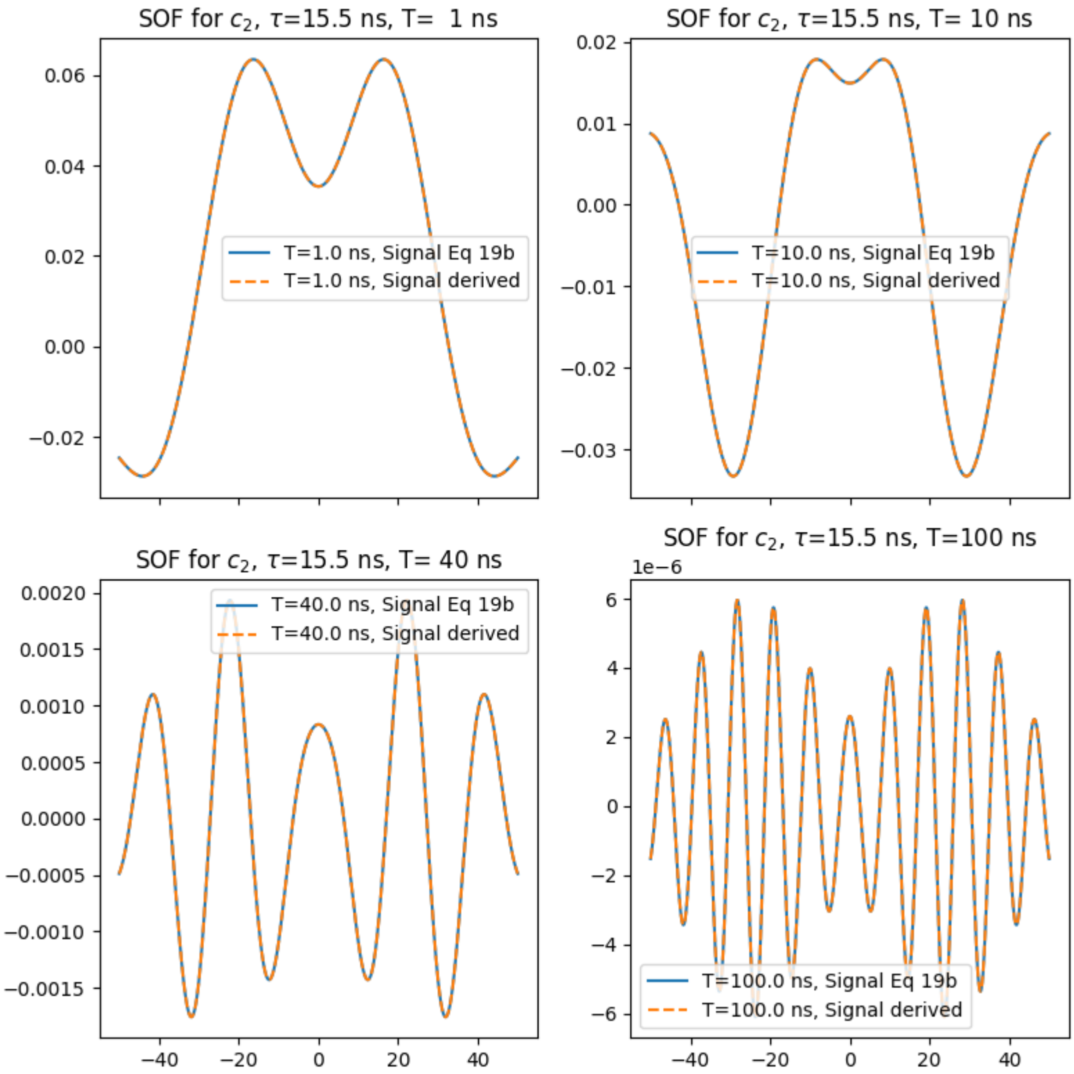
[ ]:
```
def c_2(Omega, delta, tau, T, V):
    """
    Omega: angular frequency in G rad/s
    delta: phase in rad
    tau: time in ns
    T: time in ns

    gamma_1, gamma_2 in GHz
    """
    Q = 1/2*(gamma_1 - gamma_2)

    a = np.sqrt(4*V**2+(Omega+1j*Q)**2)

    sin_theta = 2*V/a
    cos_theta = (Omega + 1j*Q)/a

    exp_part = np.exp(-1/2*(gamma_1 + gamma_2)*tau-1/2*gamma_2*T)

    sin_part = -1j*sin_theta*np.sin(1/2*a*tau)

    cos_sin_part = np.cos(1/2*a*tau)-1j*cos_theta*np.sin(1/2*a*tau)
    exp_cos_sin_part = np.exp(1/2*(gamma_2-gamma_1)*T + 1j*(delta + Omega*T)) *␣
↪\
        (np.cos(1/2*a*tau)+1j*cos_theta*np.sin(1/2*a*tau))

    return exp_part*sin_part*(cos_sin_part+exp_cos_sin_part)
```

[ ]:
```python
def c_2_modulus_square(Omega, delta, tau, T, V):
    return np.abs(c_2(Omega, delta, tau, T, V))**2
    return c_2(Omega, delta, tau, T, V)*np.conj(c_2(Omega, delta, tau, T, V))


def signal(Omega, tau, T, V):
    return np.real(c_2_modulus_square(Omega, np.pi, tau, T, V) -␣
 ↪c_2_modulus_square(Omega, 0, tau, T, V))
```

[ ]:
```python
def signal_simplified(Omega, tau, T, V):
    """
    Omega: angular frequency in G rad/s
    delta: phase in rad
    tau: time in ns
    T: time in ns

    gamma_1, gamma_2 in GHz
    """
    a = np.sqrt(4*V**2+(Omega+1j*Q)**2)
    x = (1/2)*a * tau

    cos_theta = (Omega + 1j*Q)/a
    sin_theta = 2*V/a
    exp_part = np.exp(-0.5*(gamma_1+gamma_2)*(2*tau+T))
    first_modulus_squared = np.abs(sin_theta*np.sin(x))**2

    second_modulus_squared = np.abs(np.cos(x) - 1j*cos_theta*np.sin(x))**2

    z = (np.cos(x) + 1j*cos_theta*np.sin(x)) / \
        (np.cos(x) - 1j*cos_theta*np.sin(x))

    third_modulus = np.abs(z)
    sin_part = np.cos(np.angle(z)+Omega*T)
    return -4*exp_part  *np.ones(len(Omega))␣
 ↪*first_modulus_squared*second_modulus_squared*third_modulus*sin_part
```

[ ]:
```python
tau = 15.5e-9  # s

frequency_range = 50  # MHz
frequncies = np.linspace(-frequency_range, frequency_range, 1000)  # MHz

_T_values = [1e-9, 10e-9, 40e-9, 100e-9]

num_rows = len(_T_values) // 2 + len(_T_values) % 2
num_cols = 2

fig, axs = plt.subplots(num_rows, num_cols, figsize=(8, 8), sharex=True)
```

```
for i, _T in enumerate(_T_values):
    row = i // num_cols
    col = i % num_cols
    ax = axs[row, col]
    ax.plot(frequncies, signal(2 * np.pi * frequncies * 1e6, tau, _T, V),␣
 ↪label=f"T={_T * 1e9} ns, Signal Eq 19b")
    ax.plot(frequncies, signal_simplified(2 * np.pi * frequncies * 1e6, tau,␣
 ↪_T, V), "--", label=f"T={_T * 1e9} ns, Signal derived")
    ax.legend()
    ax.set_title(f"SOF for $c_2$, $\\tau$={tau * 1e9:3g} ns, T={_T * 1e9:3g}␣
 ↪ns")

# Remove empty subplots if there are any
if len(_T_values) < num_rows * num_cols:
    for i in range(len(_T_values), num_rows * num_cols):
        fig.delaxes(axs.flatten()[i])

plt.tight_layout()
plt.show()
```


SOF for c_2, τ=15.5 ns, T= 1 ns
T=1.0 ns, Signal Eq 19b
T=1.0 ns, Signal derived
SOF for c_2, τ=15.5 ns, T= 10 ns
T=10.0 ns, Signal Eq 19b
T=10.0 ns, Signal derived
SOF for c_2, τ=15.5 ns, T= 40 ns
T=40.0 ns, Signal Eq 19b
T=40.0 ns, Signal derived
SOF for c_2, τ=15.5 ns, T=100 ns
1e−6
T=100.0 ns, Signal Eq 19b
T=100.0 ns, Signal derived

```
[ ]:
```